

Using curvature invariants for wave extraction in numerical relativity

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We present a new expression for the Weyl scalar Ψ_4 that can be used in numerical relativity to extract the gravitational wave content of a spacetime. The formula relies upon the identification of transverse tetrads, namely the ones in which $\Psi_1 = \Psi_3 = 0$. It is well known that tetrads with this property always exist in a general Petrov type I spacetime. A sub-class of these tetrads naturally converges to the Kinnersley tetrad in the limit of Petrov type D spacetime. However, the transverse condition fixes only four of the six parameters coming from the Lorentz group of transformations applied to tetrads. Here we fix the tetrad completely, in particular by giving the expression for the spin-boost transformation that was still unclear. The value of Ψ_4 in this optimal tetrad is given as a function of the two curvature invariants I and J .

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I. INTRODUCTION

With the recent breakthroughs in numerical relativity [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20] concerning the numerical evolution of binary black hole systems, the problem of extracting waveforms from a numerically evolved spacetime has become of primary importance. One of the most used techniques for wave extraction involves the Newman-Penrose formalism and in particular the calculation of the Weyl scalar Ψ_4 .

The Newman-Penrose formalism is a tetrad formalism where the tetrad vectors are chosen to be null, two of them being real (normally referred to as ℓ^μ and n^μ), the other two complex conjugated (normally referred to as m^μ and \bar{m}^μ). The idea underlying this approach is that by contracting the Weyl tensor on null vectors, the physical properties of the spacetime are easier to single out.

The relevant quantities in this formalism are the Weyl scalars, given by the contraction of the Weyl tensor over a specific combination of the four null tetrad vectors, and the connection coefficients (spin coefficients) related to the covariant derivatives of the same tetrad vectors. All of these quantities are scalars, making the choice of the coordinate system irrelevant for their calculation; they are however dependent on the tetrad choice which constitutes the gauge freedom in this formalism.

The normal equations governing the gravitational field, namely the Bianchi and Ricci identities, can be rewritten in this formalism as functions of the Weyl scalars and the spin coefficients, and a detailed presentation can be found in [21]. However, the whole set of equations is clearly higher in number than the real physical degrees of freedom, and it is not clear how these equations are related to each other.

On the other hand, it is well known that the variables introduced within this formalism, under certain assumptions, acquire a precise physical meaning. For example

Ψ_0 and Ψ_4 are related to the ingoing and outgoing gravitational wave contribution, while Ψ_2 is related to the background contribution to the curvature.

The spin coefficients can also be related to physical properties of the tetrad vectors: k and ϵ are related to the geodesic properties of the ℓ^μ vector, ρ plays a determinant role in establishing whether ℓ^μ is hypersurface orthogonal or not, and σ measures the shear of null geodesics defined by ℓ^μ .

For the mentioned physical properties to hold, a suitable tetrad choice is fundamental. The importance of a robust wave extraction technique has been underlined by some recent articles [22, 23, 24, 25]. Recent works [26, 27, 28, 29, 30, 31] have identified in *transverse tetrads*, i.e. those tetrads satisfying the condition $\Psi_1 = \Psi_3 = 0$, a convenient candidate for wave extraction and in general for a better understanding of the equations governing the Newman-Penrose formalism. Transverse tetrads can always be found in a general Petrov type I spacetime: we know in fact [32] that there are three families (frames) of transverse tetrads. The degeneracy in each single frame is due to the fact that the condition $\Psi_1 = \Psi_3 = 0$ does not fix the tetrad completely, leaving the spin-boost (type III rotation) degree of freedom yet to be specified. Therefore all the tetrads in every transverse frame can be related by a spin-boost transformation.

The advantages of the choice $\Psi_1 = \Psi_3 = 0$ have already been shown in [26, 27, 28, 29, 30, 31], namely one of these three frames naturally converges to the frame where $\Psi_0 = \Psi_4 = 0$ when the spacetime approaches Petrov type D. This property ensures that the values for the two scalars Ψ_0 and Ψ_4 , in the linear regime, are at first order tetrad invariant, and directly associated with the gravitational wave signal [33].

To determine the tetrad completely, however, one also needs to fix the spin-boost degree of freedom. The Kinnersley tetrad [34] identifies the spin-boost parameter by

imposing the condition $\epsilon = 0$. The motivation for this choice is related to the physical properties of the ℓ null vector: the geodesic equation for ℓ reads

$$\ell^\mu \nabla_\mu \ell^\nu = (\epsilon + \epsilon^*) \ell^\nu - \kappa \bar{m}^\nu - \kappa^* m^\nu, \quad (1)$$

which shows that if the two spin coefficients κ and ϵ vanish, the vector ℓ^μ is geodesic and affinely parametrized. In the limit of Kerr spacetime, the Goldberg-Sachs theorem [21] guarantees that $\kappa = 0$, so the additional condition $\epsilon = 0$ enforces the affine parametrization of ℓ^μ .

In order to find transverse frames in a numerical simulation, one procedure is to calculate the Weyl scalars using an initial tetrad, and then calculate the rotation parameters for type I and type II rotations using the two methods given in [26, 29]. This has been applied successfully in [27]. This procedure is rather lengthy to apply in practice; moreover, it does not fix the spin-boost parameter in a rigorous way. Here, we want to validate a different and simplified approach that can be used when one is not interested in the expression of the tetrad vectors, but only in the final expression for Ψ_4 in the right tetrad. This procedure gives a rigorous expression for the spin-boost parameter by enforcing the condition $\epsilon = 0$ in the Petrov type D limit, and the final result for the scalars Ψ_0 , Ψ_2 , and Ψ_4 is given as functions of the two curvature invariants I and J .

The paper is organized as follows: Sec. II presents some general definitions of the relevant quantities in the Newman-Penrose formalism and provides an equation for the three Weyl scalars Ψ_0 , Ψ_2 , and Ψ_4 that are valid in transverse frames where $\Psi_1 = \Psi_3 = 0$. Sec. III analyzes the Bianchi identities in the limit of Petrov type D spacetime in order to obtain information on the spin coefficient ϵ and its connection to the spin-boost parameter. We will show that the Bianchi identities provide only information on other spin coefficients in the limit of Petrov type D spacetime. An expression for ϵ is then obtained using the Ricci identities in Sec. IV. Finally in Sec. V we apply the result to the case of the Kinnersley tetrad by enforcing the condition $\epsilon = 0$ and obtain the corresponding spin-boost parameter value. This result leads to the final expression for the Weyl scalars in this particular tetrad.

II. GENERAL DEFINITIONS

Weyl scalars are given by contraction of the Weyl tensor over a certain combination of four null vectors, two real (ℓ^μ and n^μ) and two complex conjugates (m^μ and \bar{m}^μ), according to

$$\Psi_0 = -C_{abcd} \ell^a m^b \ell^c m^d, \quad (2a)$$

$$\Psi_1 = -C_{abcd} \ell^a n^b \ell^c m^d, \quad (2b)$$

$$\Psi_2 = -C_{abcd} \ell^a m^b \bar{m}^c n^d, \quad (2c)$$

$$\Psi_3 = -C_{abcd} \ell^a n^b \bar{m}^c n^d, \quad (2d)$$

$$\Psi_4 = -C_{abcd} n^a \bar{m}^b n^c \bar{m}^d. \quad (2e)$$

The tetrad choice constitutes the gauge degree of freedom in the calculation of Weyl scalars, and can be represented by the six parameter Lorentz group of gauge transformations. Despite Weyl scalars being tetrad dependent, it is possible to construct two quantities which are no longer dependent on the tetrad choice. Such quantities are the curvature invariants I and J , and their expressions as functions of the Weyl scalars are given by

$$I = \Psi_4 \Psi_0 - 4\Psi_1 \Psi_3 + 3\Psi_2^2, \quad (3a)$$

$$J = \det \begin{vmatrix} \Psi_4 & \Psi_3 & \Psi_2 \\ \Psi_3 & \Psi_2 & \Psi_1 \\ \Psi_2 & \Psi_1 & \Psi_0 \end{vmatrix}. \quad (3b)$$

If we assume to fix all the six degrees of freedom related to the tetrad choice by requiring that $\Psi_1 = \Psi_3 = 0$ and $\Psi_0 = \Psi_4$, the expressions for the two curvature invariants written in Eq. (3) simplify to

$$I = \Psi_4^2 + 3\Psi_2^2, \quad (4a)$$

$$J = \Psi_4^2 \Psi_2 - \Psi_2^3. \quad (4b)$$

Eq. (4a) and (4b) can now be inverted to give Ψ_2 and Ψ_4 as functions of the curvature invariants I and J . In particular one can solve for Ψ_2 by setting $\Psi_2 = \frac{\lambda}{2}$, where λ is the solution of the characteristic polynomial

$$\lambda^3 - 2I\lambda + 2J = 0. \quad (5)$$

The three possible solutions are given by

$$\lambda_1 = -\left(P + \frac{I}{3P}\right), \quad (6a)$$

$$\lambda_2 = -\left(e^{\frac{2\pi i}{3}} P + e^{\frac{4\pi i}{3}} \frac{I}{3P}\right), \quad (6b)$$

$$\lambda_3 = -\left(e^{\frac{4\pi i}{3}} P + e^{\frac{2\pi i}{3}} \frac{I}{3P}\right), \quad (6c)$$

where P is defined as

$$P = \left[J + \sqrt{J^2 - (I/3)^3} \right]^{\frac{1}{3}}. \quad (7)$$

Setting Ψ_2 equal to the three possible roots gives the three possible transverse frames. Ψ_4 is then determined accordingly once we have fixed a specific transverse frame.

Setting $\Psi_0 = \Psi_4$ for the spin-boost degree of freedom is not the best possible choice, since in this case, the two Weyl scalars have the radial fall-off of r^{-3} at future null infinity. We can nevertheless use this choice as a starting point and reinsert the spin-boost degree of freedom into the expressions for the scalars: introducing the quantity $\Theta = \sqrt{3}PI^{-\frac{1}{2}}$ and the spin-boost parameter $\mathcal{B} = \left(\frac{\Psi_4}{\Psi_0}\right)^{\frac{1}{4}}$, the three non vanishing Weyl scalars can be written as

$$\Psi_0 = -\frac{i\mathcal{B}^{-2}}{2} \cdot \Psi_-, \quad (8a)$$

$$\Psi_2 = -\frac{1}{2\sqrt{3}} \cdot \Psi_+, \quad (8b)$$

$$\Psi_4 = -\frac{i\mathcal{B}^2}{2} \cdot \Psi_-, \quad (8c)$$

where

$$\Psi_{\pm} = I^{\frac{1}{2}} \left(e^{\frac{2\pi i k}{3}} \Theta \pm e^{-\frac{2\pi i k}{3}} \Theta^{-1} \right), \quad (9)$$

and k is an integer number assuming the values $\{0, 1, 2\}$ corresponding to the three different transverse frames. For the principal root in the expression for P given in Eq. (7), the limit of type D corresponds to $\Theta \rightarrow 1$. Subsequently, from Eq. (9), the frame with $k = 0$ is the frame where Ψ_- (and consequently Ψ_0 and Ψ_4) tends to zero, i.e. the transverse frame which is also a quasi-Kinnersley frame [29].

This paper demonstrates how the expressions in Eq. (8-9) for the Weyl scalars relate to the expressions for the spin coefficients in the limit of Petrov type D spacetime; we obtain an expression for the spin coefficient ϵ and enforce the condition $\epsilon = 0$. To do this, we introduce the directional derivative operators along the null tetrad vectors $D = \ell^\mu \nabla_\mu$, $\Delta = n^\mu \nabla_\mu$, $\delta = m^\mu \nabla_\mu$ and $\delta^* = \bar{m}^\mu \nabla_\mu$, and analyze the Bianchi and Ricci identities in the Newman-Penrose formalism.

III. THE BIANCHI IDENTITIES

Setting $\Psi_1 = \Psi_3 = 0$ simplifies the Bianchi identities, given here in terms of the Weyl scalars and spin coefficients

$$D\Psi_4 = -3\lambda\Psi_2 - (4\epsilon - \rho)\Psi_4, \quad (10a)$$

$$D\Psi_2 = -\lambda\Psi_0 + 3\rho\Psi_2, \quad (10b)$$

$$\Delta\Psi_0 = 3\sigma\Psi_2 + (4\gamma - \mu)\Psi_0, \quad (10c)$$

$$\Delta\Psi_2 = \sigma\Psi_4 - 3\mu\Psi_2, \quad (10d)$$

$$\delta\Psi_4 = -3\nu\Psi_2 + (\tau - 4\beta)\Psi_4, \quad (10e)$$

$$\delta\Psi_2 = -\nu\Psi_0 + 3\tau\Psi_2, \quad (10f)$$

$$\delta^*\Psi_0 = 3\kappa\Psi_2 + (4\alpha - \pi)\Psi_0, \quad (10g)$$

$$\delta^*\Psi_2 = \kappa\Psi_4 - 3\pi\Psi_2. \quad (10h)$$

A key ingredient is rewriting the Bianchi identities in terms of the newly introduced variables Ψ_{\pm} . Since ϵ appears only in the first two Bianchi identities, we will give the details of the calculation only for the derivative operator D ; however, as the symmetry of the Bianchi identities suggests, the calculation for the other derivatives is analogous, and we will use this property at the end of this paper to calculate the expressions for the spin coefficients γ , α and β .

Using Eq. (8) to relate the Weyl scalars Ψ_0 , Ψ_2 and Ψ_4 to the new scalars Ψ_+ and Ψ_- one obtains

$$D\Psi_+ = -\tilde{\lambda}\Psi_- + 3\rho\Psi_+, \quad (11a)$$

$$D\Psi_- = \tilde{\lambda}\Psi_+ - (4\tilde{\epsilon} - \rho)\Psi_-, \quad (11b)$$

$$\Delta\Psi_+ = \tilde{\sigma}\Psi_- - 3\mu\Psi_+, \quad (11c)$$

$$\Delta\Psi_- = -\tilde{\sigma}\Psi_+ + (4\tilde{\gamma} - \mu)\Psi_-, \quad (11d)$$

$$\delta\Psi_+ = -\tilde{\nu}\Psi_- + 3\tau\Psi_+, \quad (11e)$$

$$\delta\Psi_- = \tilde{\nu}\Psi_+ - (4\tilde{\beta} - \tau)\Psi_-, \quad (11f)$$

$$\delta^*\Psi_+ = \tilde{\kappa}\Psi_- - 3\pi\Psi_+, \quad (11g)$$

$$\delta^*\Psi_- = -\tilde{\kappa}\Psi_+ + (4\tilde{\alpha} - \pi)\Psi_-, \quad (11h)$$

where we have introduced the rescaled spin coefficients $\tilde{\lambda} = i\sqrt{3}\lambda\mathcal{B}^{-2}$, $\tilde{\sigma} = i\sqrt{3}\sigma\mathcal{B}^2$, $\tilde{\nu} = i\sqrt{3}\nu\mathcal{B}^{-2}$, $\tilde{\kappa} = i\sqrt{3}\kappa\mathcal{B}^2$, $\tilde{\epsilon} = \epsilon + \frac{1}{2}D\ln\mathcal{B}$, $\tilde{\gamma} = \gamma + \frac{1}{2}\Delta\ln\mathcal{B}$, $\tilde{\beta} = \beta + \frac{1}{2}\delta\ln\mathcal{B}$, $\tilde{\alpha} = \alpha + \frac{1}{2}\delta^*\ln\mathcal{B}$.

This new set of rescaled spin coefficients now transforms in the same way under a spin-boost transformation: for example the three spin coefficients $\{\rho, \tilde{\lambda}, \tilde{\epsilon}\}$ transform as $\rho \rightarrow |\mathcal{B}|^{-1}\rho$, $\tilde{\epsilon} \rightarrow |\mathcal{B}|^{-1}\tilde{\epsilon}$ and $\tilde{\lambda} \rightarrow |\mathcal{B}|^{-1}\tilde{\lambda}$, and analogous transformations for the other spin coefficients. This is not surprising: Ψ_+ and Ψ_- are only functions of curvature invariants, so in Eq. (11a) and (11b) the only dependence on the spin-boost parameter on the left hand side comes from the D derivative operator which carries a $|\mathcal{B}|^{-1}$ factor, the right hand side must therefore be consistent and show the same spin-boost dependence in the rescaled spin coefficients.

Dividing Eq. (11a) by Ψ_+ and Eq. (11b) by Ψ_- the first two Bianchi identities become:

$$\frac{D\Psi_+}{\Psi_+} = -\tilde{\lambda}\frac{\Psi_-}{\Psi_+} + 3\rho, \quad (12a)$$

$$\frac{D\Psi_-}{\Psi_-} = \tilde{\lambda}\frac{\Psi_+}{\Psi_-} - (4\tilde{\epsilon} - \rho). \quad (12b)$$

We will now study the behavior of Eq. (12) in the Petrov type D limit. Using Eq. (9) and applying the D operator to Ψ_+ and Ψ_- gives

$$D\Psi_+ = D\ln\Theta \cdot \Psi_- + D\ln\left(I^{\frac{1}{2}}\right)\Psi_+, \quad (13a)$$

$$D\Psi_- = D\ln\Theta \cdot \Psi_+ + D\ln\left(I^{\frac{1}{2}}\right)\Psi_-. \quad (13b)$$

In the Petrov type D limit (corresponding to $\Theta \rightarrow 1$) one has $D\Psi_+ \rightarrow D \ln \left(I^{\frac{1}{2}}\right) \Psi_+$ and $D\Psi_- \rightarrow D \ln \left(I^{\frac{1}{2}}\right) \Psi_-$. This result implies that the left hand sides in Eq. (12) tend to the same value, i.e. $D \ln \left(I^{\frac{1}{2}}\right)$, and therefore also the right hand sides can be set to be equal in this limit. Moreover, the ratio $\frac{\Psi_-}{\Psi_+} \rightarrow -i \tan \left(\frac{2\pi k}{3}\right)$ in the same limit. Putting this all together, and subtracting Eq. (12b) from Eq. (12a) we find that the following relation between spin coefficients holds in the Petrov type D limit

$$(\rho + 2\tilde{\epsilon}) \sin \left(\frac{4\pi k}{3}\right) + i\tilde{\lambda} \cos \left(\frac{4\pi k}{3}\right) = 0. \quad (14)$$

Eq. (14) is valid for all three transverse frames, depending on the value of k . If we assume to be in the transverse frame that is also a quasi-Kinnersley frame, which corresponds to having $k = 0$, Eq. (14) reduces to $\tilde{\lambda} = 0$, consistently with the Goldberg-Sachs theorem. One can then use Eq. (11a) to find the expression for ρ , obtaining $\rho = D \ln I^{\frac{1}{2}}$, but the key point here is that in the quasi-Kinnersley frame the Bianchi identities leave the expression for $\tilde{\epsilon}$ completely unresolved. However, it is really the expression for $\tilde{\epsilon}$ we are interested in, as it is the one related to the spin-boost transformation. To obtain additional information on this spin coefficient, we therefore analyze the Ricci identities.

IV. THE RICCI IDENTITIES

In this section, we will use the Ricci identities to understand how the spin coefficients ϵ , γ , α and β relate to the spin-boost parameter \mathcal{B} . We will first show that they can be expressed as directional derivatives of the same function, and then determine the equation that this function must satisfy in the limit of Petrov type D.

A. Spin coefficients as directional derivatives

We assume to be in the Petrov type D limit, where $\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$ and also, as a consequence of the Goldberg-Sachs theorem, the four spin coefficients λ , σ , ν and κ are vanishing. We begin with the following Ricci identity, obtained after adding and subtracting the product $\beta\epsilon$ on the right-hand side:

$$D\beta - \delta\epsilon = \epsilon(\pi^* - \alpha^* - \beta) + \beta(\rho^* + \epsilon - \epsilon^*). \quad (15)$$

Introducing the rescaled spin coefficients, one can re-express this Ricci identity in terms of $\tilde{\epsilon}$ and $\tilde{\beta}$, leading to

$$D\tilde{\beta} - \delta\tilde{\epsilon} = \tilde{\epsilon}(\pi^* - \alpha^* - \beta) + \tilde{\beta}(\rho^* + \epsilon - \epsilon^*). \quad (16)$$

Comparing Eq. (15) with the expression of the commutator $[D, \delta]$ (again assuming $\sigma = \kappa = 0$)

$$[D, \delta] = (\pi^* - \alpha^* - \beta)D + (\rho^* + \epsilon - \epsilon^*)\delta, \quad (17)$$

it is possible to see that the Ricci identity is consistent with having $\tilde{\epsilon} = D\mathcal{H}_1$ and $\tilde{\beta} = \delta\mathcal{H}_1$, where \mathcal{H}_1 is a function to be determined. Using the equivalent Ricci identity obtained after exchanging the tetrad vectors $\ell \leftrightarrow n$ and $m \leftrightarrow \bar{m}$

$$\Delta\tilde{\alpha} - \delta^*\tilde{\gamma} = \tilde{\alpha}(\gamma^* - \gamma - \mu^*) + \tilde{\gamma}(\alpha + \beta^* - \tau^*), \quad (18)$$

we obtain an equivalent result for the spin coefficients $\tilde{\gamma}$ and $\tilde{\alpha}$ and conclude that they also can be expressed as $\tilde{\gamma} = \Delta\mathcal{H}_2$ and $\tilde{\alpha} = \delta^*\mathcal{H}_2$.

Using the properties of transformation of the spin coefficients under the exchange operation $\ell^\mu \leftrightarrow n^\mu$ and $m^\mu \leftrightarrow \bar{m}^\mu$, i.e. $\tilde{\epsilon} \leftrightarrow -\tilde{\gamma}$ and $\tilde{\alpha} \leftrightarrow -\tilde{\beta}$, we conclude that $\mathcal{H}_1 = -\mathcal{H}_2 = \mathcal{H}$. The four spin coefficients can then be written as $\tilde{\epsilon} = D\mathcal{H}$, $\tilde{\gamma} = -\Delta\mathcal{H}$, $\tilde{\beta} = \delta\mathcal{H}$ and $\tilde{\alpha} = -\delta^*\mathcal{H}$.

The original spin coefficients are therefore given by

$$\epsilon = D\mathcal{H} - \frac{1}{2}D \ln \mathcal{B} = D\mathcal{H}_-, \quad (19a)$$

$$\gamma = -\Delta\mathcal{H} - \frac{1}{2}\Delta \ln \mathcal{B} = -\Delta\mathcal{H}_+, \quad (19b)$$

$$\beta = \delta\mathcal{H} - \frac{1}{2}\delta \ln \mathcal{B} = \delta\mathcal{H}_-, \quad (19c)$$

$$\alpha = -\delta^*\mathcal{H} - \frac{1}{2}\delta^* \ln \mathcal{B} = -\delta^*\mathcal{H}_+, \quad (19d)$$

where $\mathcal{H}_\pm = \mathcal{H} \pm \frac{1}{2} \ln \mathcal{B}$. We can now use some of the remaining Ricci identities to find the expression for \mathcal{H} .

B. The function \mathcal{H}

We consider the two following Ricci identities

$$D\gamma - \Delta\epsilon = \alpha(\tau + \pi^*) + \beta(\tau^* + \pi) + \tau\pi \quad (20a)$$

$$- \gamma(\epsilon + \epsilon^*) - \epsilon(\gamma + \gamma^*) + \Psi_2,$$

$$\delta\alpha - \delta^*\beta = \mu\rho + \alpha\alpha^* + \beta\beta^* - 2\alpha\beta \quad (20b)$$

$$+ \gamma(\rho - \rho^*) + \epsilon(\mu - \mu^*) - \Psi_2.$$

The rescaled spin coefficients remove the spin-boost dependence in these identities, giving

$$D\tilde{\gamma} - \Delta\tilde{\epsilon} = \tilde{\alpha}(\tau + \pi^*) + \tilde{\beta}(\tau^* + \pi) + \tau\pi \quad (21a)$$

$$- \tilde{\gamma}(\epsilon + \epsilon^*) - \tilde{\epsilon}(\gamma + \gamma^*) + \Psi_2,$$

$$\delta\tilde{\alpha} - \delta^*\tilde{\beta} = \mu\rho + \tilde{\alpha}\alpha^* + \tilde{\beta}\beta^* - 2\tilde{\alpha}\tilde{\beta} \quad (21b)$$

$$+ \tilde{\gamma}(\rho - \rho^*) + \tilde{\epsilon}(\mu - \mu^*) - \Psi_2.$$

Having just found that the reduced spin coefficients on the left-hand sides can be expressed as directional derivatives of \mathcal{H} , one can use the definition of double derivatives in the Newman-Penrose formalism to find an equivalent form of Eq. (20). In particular the equations we will make use of are the following

$$D\Delta = -(\epsilon + \epsilon^*)\Delta + \pi\delta + \pi^*\delta^* + \ell^\mu n^\nu \nabla_\mu \nabla_\nu, \quad (22a)$$

$$\Delta D = (\gamma + \gamma^*)D - \tau^*\delta - \tau\delta^* + n^\mu \ell^\nu \nabla_\mu \nabla_\nu, \quad (22b)$$

$$\delta\delta^* = \mu D - \rho^*\Delta - (\beta - \alpha^*)\delta^* + m^\mu \bar{m}^\nu \nabla_\mu \nabla_\nu, \quad (22c)$$

$$\delta^*\delta = \mu^*D - \rho\Delta + (\alpha - \beta^*)\delta + \bar{m}^\mu m^\nu \nabla_\mu \nabla_\nu. \quad (22d)$$

As an example, we calculate the term $D\tilde{\gamma}$ on the left hand side of Eq. (21a). Using the property just found that in the Petrov type D limit $\tilde{\gamma} = -\Delta\mathcal{H}$, this term is given by $D\tilde{\gamma} = -D\Delta\mathcal{H}$, and using Eq. (22a) this corresponds to

$$\begin{aligned} -D\Delta\mathcal{H} &= (\epsilon + \epsilon^*)\Delta\mathcal{H} - \pi\delta\mathcal{H} - \pi^*\delta^*\mathcal{H} \\ &\quad - \ell^\mu n^\nu \nabla_\mu \nabla_\nu \mathcal{H}. \end{aligned} \quad (23)$$

Substituting $\tilde{\alpha} = -\delta^*\mathcal{H}$ and $\tilde{\beta} = \delta\mathcal{H}$ gives

$$D\tilde{\gamma} = -(\epsilon + \epsilon^*)\tilde{\gamma} - \pi\tilde{\beta} + \pi^*\tilde{\alpha} - \ell^\mu n^\nu \nabla_\mu \nabla_\nu \mathcal{H}. \quad (24)$$

Repeating the same procedure for $\Delta\tilde{\epsilon}$, $\delta\tilde{\alpha}$ and $\delta^*\tilde{\beta}$, and comparing with the Ricci identities in Eq. (21), one finds the two following identities

$$2\ell^\mu n^\nu \nabla_\mu \nabla_\nu \mathcal{H} = -2\pi\tilde{\beta} - 2\tau\tilde{\alpha} - \pi\tau - \Psi_2, \quad (25a)$$

$$2m^\mu \bar{m}^\nu \nabla_\mu \nabla_\nu \mathcal{H} = -2\mu\tilde{\epsilon} - 2\rho\tilde{\gamma} - \mu\rho + \Psi_2. \quad (25b)$$

Subtracting Eq. (25b) from Eq. (25a), and using the expression for the metric $g^{\mu\nu} = 2\ell^{(\mu}n^{\nu)} - 2m^{(\mu}\bar{m}^{\nu)}$, it is possible to obtain the final equation for \mathcal{H} :

$$\nabla^\mu \nabla_\mu \mathcal{H} + \nabla^\mu \ln \left(I^{\frac{1}{6}} \right) \nabla_\mu \left(2\mathcal{H} + \ln I^{\frac{1}{12}} \right) = -2\Psi_2, \quad (26)$$

where we have also used the fact that in the Petrov type D limit $\rho = D \ln I^{\frac{1}{6}}$, $\mu = -\Delta \ln I^{\frac{1}{6}}$, $\tau = \delta \ln I^{\frac{1}{6}}$ and $\pi = -\delta^* \ln I^{\frac{1}{6}}$.

In the next section we will solve Eq. (26) for the single black hole case to obtain the condition on the spin-boost parameter.

V. THE KERR LIMIT

We can now apply the results we just found to the particular case of the Kerr solution using Boyer-Lindquist coordinates. The metric in this case reads

$$\begin{aligned} ds^2 &= \left(1 - \frac{2Mr}{\Sigma} \right) dt^2 + \left(\frac{4Mar \sin^2 \theta}{\Sigma} \right) dt d\phi - \left(\frac{\Sigma}{\Gamma} \right) dr^2 \\ &\quad - \Sigma d\theta^2 - \sin^2 \theta \left(\frac{r^2 + a^2 + 2Mar \sin^2 \theta}{\Sigma} \right) d\phi^2, \end{aligned} \quad (27)$$

where $\Gamma = r^2 - 2Mr + a^2$ (in the usual notation this quantity is referred to as Δ , but here, we changed notation to avoid confusion with the derivative operator Δ), $\Sigma = r^2 + a^2 \cos^2 \theta$, M is the black hole mass and a its rotation parameter.

The Kinnersley tetrad in this coordinate system is given by

$$\ell^\mu = [(r^2 + a^2)/\Gamma, 1, 0, a/\Gamma], \quad (28a)$$

$$n^\mu = [r^2 + a^2, -\Gamma, 0, a] / (2\Sigma), \quad (28b)$$

$$m^\mu = [ia \sin \theta, 0, 1, i/\sin \theta] / \sqrt{2}\bar{\rho}, \quad (28c)$$

where $\bar{\rho} = r + ia \cos \theta$. The solution for Eq. (26) in this particular coordinate system reads

$$\mathcal{H} = \frac{1}{2} \ln \left(\Gamma^{\frac{1}{2}} I^{\frac{1}{6}} \sin \theta \right). \quad (29)$$

We now have all the elements to find the values of the spin coefficients ϵ , γ , β and α in the limit of type D, and in particular the condition on the spin-boost parameter. As already shown, the four spin coefficients ϵ , γ , α and β can be written as follows

$$\epsilon = D\mathcal{H} - \frac{1}{2}D \ln \mathcal{B}, \quad (30a)$$

$$\gamma = -\Delta\mathcal{H} - \frac{1}{2}\Delta \ln \mathcal{B}, \quad (30b)$$

$$\beta = \delta\mathcal{H} - \frac{1}{2}\delta \ln \mathcal{B}, \quad (30c)$$

$$\alpha = -\delta^*\mathcal{H} - \frac{1}{2}\delta^* \ln \mathcal{B}. \quad (30d)$$

This result can be compared with the expressions for the same spin coefficients in the Kinnersley tetrad, given by

$$\epsilon = 0, \quad (31a)$$

$$\gamma = \mu + \rho\rho^* (r - M)/2, \quad (31b)$$

$$\beta = \cot \theta / (2\sqrt{2}\bar{\rho}), \quad (31c)$$

$$\alpha = \pi - \beta^*. \quad (31d)$$

Let us consider first the spin coefficient ϵ . Using Eq. (30a) and the solution for \mathcal{H} found in Eq. (29) we can rewrite ϵ in the following way

$$\epsilon = \frac{1}{2}D \ln \left(\Gamma^{\frac{1}{2}} I^{\frac{1}{6}} \mathcal{B}^{-1} \sin \theta \right). \quad (32)$$

In order for this expression to be zero, the function inside the logarithm must be constant with respect to the derivative operator D . Given the form of the Kinnersley tetrad in Eq. (28), one concludes that the D operator corresponds to the simple ∂_r derivative (assuming that the functions do not have a t or ϕ dependence, which is indeed the case, as the Kerr spacetime is stationary and axisymmetric). As a consequence of this, ϵ vanishes if the function on the right hand side is a generic function only of the coordinate θ . This leads to the following condition on the spin-boost parameter

$$\mathcal{B} = \mathcal{B}_0 f(\theta) I^{\frac{1}{6}} \Gamma^{\frac{1}{2}} \sin \theta, \quad (33)$$

where \mathcal{B}_0 is an integration constant. It can be easily shown that the spin coefficient γ given in Eq. (31b) is consistent with Eq. (33), imposing no further condition on $f(\theta)$.

The spin coefficient β can instead be used to find the unknown function $f(\theta)$: the derivative operator δ is given by $\frac{1}{\sqrt{2\rho}}\partial_\theta$ and is therefore related to the θ dependence of the spin-boost parameter. A straightforward calculation gives $f(\theta) = \sin^{-1}\theta$, consistent also with the spin coefficient α . The final result for \mathcal{B} reads

$$\mathcal{B} = \mathcal{B}_0 I^{\frac{1}{6}} \Gamma^{\frac{1}{2}}, \quad (34)$$

When applied to the expression for the Weyl scalars given in Eq. (8), Eq. (34) gives

$$\Psi_0 = \mathcal{B}_0^{-2} \cdot \Gamma^{-1} I^{\frac{1}{6}} (\Theta - \Theta^{-1}), \quad (35a)$$

$$\Psi_2 = -\frac{1}{2\sqrt{3}} \cdot I^{\frac{1}{2}} (\Theta + \Theta^{-1}), \quad (35b)$$

$$\Psi_4 = \mathcal{B}_0^2 \cdot \Gamma I^{\frac{5}{6}} (\Theta - \Theta^{-1}). \quad (35c)$$

It is remarkable how these expressions for the scalars immediately give the correct radial fall-offs at future null infinity once the peeling behavior of the Weyl tensor is assumed: the function Γ is only defined in the limit of Petrov type D and gives no radial contribution at future null infinity; we find the same result for Θ as it is the ratio of quantities that have the same radial behavior at future null infinity. In conclusion, the quantities that give a contribution at future null infinity are the factors $I^{\frac{1}{6}}$, $I^{\frac{1}{2}}$ and $I^{\frac{5}{6}}$; given that under the peeling assumption $I \propto r^{-6}$, this corresponds to $\Psi_0 \propto r^{-1}$, $\Psi_2 \propto r^{-3}$ and $\Psi_4 \propto r^{-5}$.

The fact that we obtain radial fall-offs for Ψ_0 and Ψ_4 that are exchanged with respect to the normal assumption of outgoing radiation (where $\Psi_0 \propto r^{-5}$ and $\Psi_4 \propto r^{-1}$) is not surprising: this is due to the fact that in the Kinnersley tetrad the null vector ℓ^μ is ingoing while n^μ is outgoing. The normal assumption requires instead the opposite situation where ℓ^μ is outgoing and n^μ is ingoing. This means that one needs to exchange $\ell^\mu \leftrightarrow n^\mu$ to have the right convention. This results in $\mathcal{B} \rightarrow \mathcal{B}^{-1}$ and the Weyl scalars are changed to

$$\Psi_0 = \mathcal{B}_0^2 \cdot \Gamma I^{\frac{5}{6}} (\Theta - \Theta^{-1}), \quad (36a)$$

$$\Psi_2 = -\frac{1}{2\sqrt{3}} \cdot I^{\frac{1}{2}} (\Theta + \Theta^{-1}), \quad (36b)$$

$$\Psi_4 = \mathcal{B}_0^{-2} \cdot \Gamma^{-1} I^{\frac{1}{6}} (\Theta - \Theta^{-1}). \quad (36c)$$

giving this time, as expected, the correct radial fall-offs for Ψ_0 and Ψ_4 .

Eqs. (36) are the main result that we propose for wave extraction in numerical relativity. As evident from the equations, the conditions on the spin coefficients do not completely fix the values of the Weyl scalars, leaving the complex constant \mathcal{B}_0 undetermined. This is not surprising as such conditions involve the directional derivatives along the tetrad null vectors and are therefore independent of additional constant multiplication factors. The optimal value of this integration constant will have to be determined enforcing the values of the spin coefficients ρ , μ , τ and π ; the result of this calculation will be presented in a following numerical paper. We are also investigating the comparison of these expressions with the analogous quantities defined in the characteristic formulation of Einstein's equations [22, 24, 25]. As we expect, this should give us more insights on how to choose this integration constant from a theoretical point of view. This is the subject of future work on this topic.

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